MTH 303
Real analysis **Homework 5**

Notation: E denotes a nonempty subset of $\mathbb{R}$ unless specified otherwise.

1. Let p be a limit point of E. If f and g are real-valued functions defined on E such that
$$\lim_{x\to p} f(x) = A \quad \text{and} \quad \lim_{x\to p} g(x) = B$$
then show that
   (a) $\lim_{x\to p}(f+g)(x) = A + B$
   (b) $\lim_{x\to p}(fg)(x) = AB$
   (c) $\lim_{x\to p}(\frac{f}{g})(x) = \frac{A}{B}$, if $B \neq 0$

2. Given two functions $f : E \to \mathbb{R}$ and $g : f(E) \to \mathbb{R}$ be functions, consider the function $h : E \to \mathbb{R}$ defined by $h(x) = g(f(x)),\ x \in E$. If f is continuous at $p \in E$ and g is continuous at $f(p) \in f(E)$, then show that h is continuous at p.

3. A function $f : E \to \mathbb{R}$ is continuous if and only if $f^{-1}(C)$ is closed for every closed set C in $\mathbb{R}$.

4. Suppose E is a compact set and $f : E \to \mathbb{R}$ is a continuous and one-one function on E, then the inverse function f^{-1} defined on $f(E)$ by $f^{-1}(f(x)) = x,\ x \in E$, is continuous on $f(E)$.

5. Let f be a continuous real-valued function defined on the interval $[a, b]$. If $f(a) < f(b)$ and if $c \in (f(a), f(b))$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

6. Suppose f is a real-valued function defined on $\mathbb{R}$ such that
$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$$
for every point $x \in \mathbb{R}$. Does this imply that f is continuous?

7. Let f be a real-valued continuous function on $\mathbb{R}$. Show that $f(\bar{E}) \subseteq \overline{f(E)}$ for every subset E in $\mathbb{R}$. Also, show by an example, that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.

8. Let $f : E \to \mathbb{R}$ be a continuous function and $Z(f)$ denote the set of all points $p \in E$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.
The set $Z(f)$ is said to be the zero set of f.

9. Let f and g be real-valued continuous functions defined on $\mathbb{R}$ such that $f(p) = g(p)$ for all $p \in \mathbb{Q}$. Prove that $f(p) = g(p)$ for all $p \in \mathbb{R}$.

10. If f is a real-valued continuous function defined on a closed set $E \subseteq \mathbb{R}$, prove that there exists a continuous real-valued function g defined on $\mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$. (Such a function g is called a continuous extension of f from E to $\mathbb{R}$.) Show that the result becomes false if the assumption that E is closed is dropped.

11. Let f be a real-valued uniformly continuous function on a bounded set E in $\mathbb{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted.

12. If f is not uniformly continuous then show that for some $\epsilon > 0$ there exist sequences $\{p_n\}$ and $\{q_n\}$ such that $|p_n - q_n| \to 0$ but $|f(p_n) - f(q_n)| \geq \epsilon$.

13. Show that uniform continuous functions map Cauchy sequences to Cauchy sequences, i.e., if f is uniformly continuous and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous function from I to itself. Prove that there exists a point $x \in I$ such that $f(x) = x$. (Such a point x is called a fixed point of f.)

15. Let f be a real-valued continuous function defined on $\mathbb{R}$ such that $f(v)$ is open for every open set V in $\mathbb{R}$. Prove that f is monotonic.

16. Let $(x) = x - [x]$ denote the fractional part of x, where $[x]$ denote the largest integer contained in x. What discontinuities do the functions $x \to (x)$ and $x \to [x]$ have?

17. Consider the function f defined on $\mathbb{R}$ by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$, where m and n are integers without any common divisors. Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

18. Define the distance from $x \in \mathbb{R}$ to E by

$$\rho_E(x) = \inf_{y \in E} |x - y|.$$

   (a) Prove that $\rho_E(x) = 0$ if and only if $x \in \bar{E}$.

   (b) Prove that ρ_E is uniformly continuous function on $\mathbb{R}$, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R}.$$

19. Let K and F are disjoint sets in $\mathbb{R}$. If K is compact and F is closed, then prove that there exists a $\delta > 0$ such that $|x - y| > \delta$ if $x \in K$ and $y \in F$. Show that the conclusion may fail for two disjoint closed sets if neither is compact.

20. Let A and B be disjoint nonempty closed sets in $\mathbb{R}$. Define

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}, \; x \in \mathbb{R},$$

where $\rho_A(x)$ denotes the distance from $x \in \mathbb{R}$ to A as defined in Question 18. Show that f is a continuous function on $\mathbb{R}$ whose range lies in the closed unit interval $[0, 1]$. Also, show that $f(x) = 0$ precisely on A and $f(x) = 1$ precisely on B.